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LONGITUDINAL MIXING MEASURED BY RADIOACTIVE TRACERS

By Harold A. Thomas, Jr., and Ralph S. Archibald, Jun. ASCE

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PAPERS

LONGITUDINAL MIXING MEASURED BY RADIOACTIVE TRACERS

By Harold A. Thomas, Jr.,[1] and Ralph S. Archibald,[2] Jun. ASCE

Synopsis

A method of determining the magnitude and effect of horizontal mixing in pipes and tanks is available to the engineer through the use of radioactive tracers. These materials have shown many advantages over the commonly accepted use of salts and dyes.

The foremost advantage of radioisotopes is the ability to detect them in minute concentrations, eliminating misleading density currents that may occur when brine is used. In addition, precise methods of detection are available, and in some instances it is not necessary to take samples of the fluid being tested.

The paper developes a longitudinal mixing parameter that characterizes a given system and gives examples of its use. The experimental results are reported and analyzed, and techniques are briefly discussed.

Introduction

Salts and dyes have been used as tracers in flow systems primarily for the measurement of discharge. In the Allen[3] salt-velocity method, a slug or cloud of salt solution is abruptly injected into the flow, and a measurement is made of the time taken by the salt slug to travel to a downstream station. In sanitary engineering research, this technique has another and more important application, namely, the measuring of longitudinal mixing.

Part I—Theory of Longitudinal Mixing

The phenomenon of "shortcircuiting" of flow in the passage of fluids through conduits, streams, tanks, and lakes is the result of three factors: (1) The non-

Note.—Written comments are invited for publication; the last discussion should be submitted by February 1, 1952.

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[2] Engr., Pipe Founders Sale Corp.

[3] "The Salt Velocity Method of Water Measurement," by Charles M. Allen and Edwin A. Taylor *Transactions*, American Society of Mechanical Engineers, Vol. 45, 1923, p. 285.

uniform distribution of velocity caused by the curvature of and the shear at the solid boundaries of the flow; (2) eddy diffusion engendered by the turbulence of the moving stream; and (3) irregular displacements due to wind, local density differences, sludge removal mechanisms, boats, and pumps. The mixing from these sources causes the tracer-cloud to disperse as it moves downstream. This action may be depicted diagrammatically for a one-dimensional flow system in which the origin represents the point of injection of the tracer, as shown in Fig. 1.

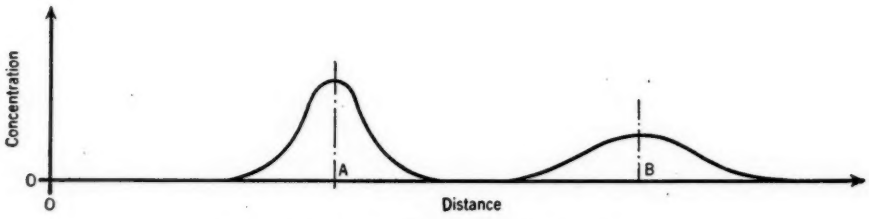

FIG. 1.—SPACE DISTRIBUTION OF TRACER CLOUD

In some flows, such as those occurring in settling tanks and in ponds, the mixing action causes portions of the fluid to travel to a downstream station with velocities many times larger than the average velocity. The time required by different portions of the flow to travel to downstream stations may be measured by observing the rate of arrival of the dispersing cloud of tracer. This may be shown in two types of diagrams depicted in Fig. 2. The concentration-time

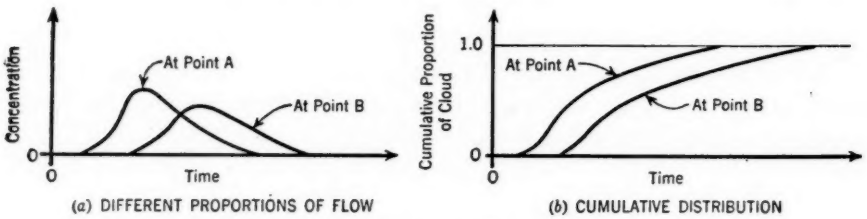

FIG. 2.—TIME OF TRAVEL OF SALT CLOUD

distribution is generally asymmetrical because the slug continues to disperse during the period that it is passing the observation station.

THE EFFECT OF LONGITUDINAL MIXING

In comparing the degree of longitudinal mixing in different flow systems it is convenient to refer the actual flow time to the average time of flow. Average flow time (T) is generally taken as the arithmetic mean time of flow that may be calculated for unsteady flow from the formula

$$T = \frac{\int_0^{t_1} t\,Q\,dt}{\int_0^{t_1} Q\,dt} \qquad (1)$$

in which Q is the variable discharge, and t_1 is the time for which the flow is averaged. For steady flow the equation becomes

$$T = \frac{V}{Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which V is the volume of the fluid between the two reference stations. The average flow time is the abscissa of the centroid of the time-concentration distribution (Fig. 2(*a*)).

The importance of longitudinal mixing is due to the fact that many dangerous and objectionable substances in water, such as pathogenic bacteria, human excrement, putrescible settleable solids, and radioactive wastes are destroyed (or removed) at rates approximately proportional to their concentrations. In still water the rate of disappearance (or decay or die-away) of such substances may be calculated approximately from the formula

$$u = u_0 e^{-kt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which u is the concentration at time t (u_0 representing the initial concentration), k is a constant that may readily be determined experimentally, and e is the Napierian base of logarithms. The more complicated case of the die-away curve in turbulent flow is analyzed in the appendix.

When wastes of these types are faced in streams, pipes, or tanks, the downstream concentrations may be considerably higher than those given by Eq. 3 when t is taken as the average flow time. This is caused by short-circuiting in the flow. The efficiency of sewage treatment, water purification, and disinfection systems, and the capacity of streams and ponds for handling pollutional loads depends to a large extent upon the amount of longitudinal mixing occurring in the flow. The greater the amount of mixing, the higher the concentrations found downstream; laboratory tests performed in still water as a rule yield lower downstream concentrations than those attainable in moving water.

Some idea as to the possible magnitude of the short-circuiting effect may be had from an example. Assume a chlorine contact tank for the protection of a bathing beach has a detention period of 10 min and handles settled sewage containing 1,000,000 coliform organisms per milliliter. The tank is provided with 9 transverse baffles. Laboratory tests in still water determined that coliform organisms are destroyed by the chlorine in accordance with Eq. 3 at a rate corresponding to a k of 1.0 per min. The density of bacteria in the effluent may be calculated (*a*) by assuming that no mixing takes place, that is, the fluid moves as a unit, or (*b*) by assuming that violent mixing occurs and that the bacterial concentration in the tank in all regions is uniform.

In accordance with assumption (*a*) the coliform density in the effluent would be

$$u_1 = 10^6 e^{-10} = 50 \text{ coliform per ml}$$

In accordance with assumption (*b*) the density would be given approximately by the expression $u_2 = \frac{10^6}{10}\left[e^{-0.5} + e^{-1.5} + \cdots + e^{-9.5}\right]$ in which it is assumed that as a result of the violent mixing the effluent consists of ten portions having average detention times of 0.5, 1.5, $\cdots$ 9.5 min each. A more accurate expression is given by the equation $u_2 = \frac{10^6}{10}\int_0^{10} e^{-t}\,dt = -\frac{10^6}{10}\left[e^{-t}\right]_0^{10} = 10^5$ coliform per ml. The effect of longitudinal mixing in this case causes a bacterial density in the effluent of 2,000 times that occurring with no mixing. Actually the tank would give a performance between these limiting values. However, many existing tanks tend to perform more nearly in accordance with the second assumption than the first, and longitudinal mixing may seriously interfere with the efficiency of disinfection.

Coefficient of Quiescence

The mixing performance of a tank or stream may be anticipated from the concentration-time distribution characteristics of the system obtained in tracer tests. The average concentration ($\bar{u}$) of a foreign substance at a downstream station in steady flow may be computed by the formula

$$\bar{u} = \int_0^\infty u\,p\,dt \qquad (4)$$

in which $p\,dt$ represents the proportion of the flow (ordinate of Fig. 2(*a*) or slope of Fig. 2(*b*)) passing the observation point during the time interval dt. The value of $\bar{u}$ can be found by Eq. 3. The average concentration ($\bar{u}$) may be computed by an arithmetic integration of Eq. 4, or it may be approximated by a simpler procedure. Many time-concentration distributions, it has been found, may be fitted satisfactorily by the equation

$$p\,dt = \frac{1\,N^N}{T\,(N-1)!}\left(\frac{t}{T}\right)^{N-1} e^{-\frac{Nt}{T}}\,dt \qquad (5)$$

in which N is the coefficient of quiesence, a parameter that depends upon the degree of longitudinal mixing. For flow systems in which the mixing is very violent N approaches a limiting lower value of 1.0; for systems in which the mixing is zero (uniform, irrotational flow) N equals infinity. Test results for ordinary tanks and ponds show N-values range from 1.2 to about 6, depending upon the shape, baffling, inlet and outlet arrangements of the tank and the amount of stirring induced by wind, density differences, and sludge removal mechanisms.

A number of different procedures may be used to evaluate N from a particular tracer time-concentration distribution. The simplest method of fitting Eq. 5 consists of making the means and modes of the theoretical and observed curves equal. This leads to the relation

$$N = \frac{\text{mean time of flow}}{\text{mean time of flow—modal time of flow}} \qquad (6)$$

The mean time of flow is the centroid of the distribution, T, and the modal time is the time corresponding to the peak concentration.

Integrating Eq. 4 by using the values of u and p as given by Eq. 4 and Eq. 5, the following relation is obtained:

$$\bar{u} = u_0 \left(1 + \frac{k\,T}{N} \right)^{-N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Values of the relative proportion of the concentration $\left(\frac{\bar{u}}{u_0}\right)$ remaining in the flow as it passes the downstream section, for various values of $k\,T$ and N, are shown in Table I.

TABLE I.—Percentage of Initial Concentration Remaining in Flow After Passage to Downstream Section

$\frac{k\,T}{N}$	Values of coefficient of quiescence, N				
	1	2	4	6	10
0	100	100	100	100	100
1	50	25	6.25	1.56	0.98
2	33.3	11.1	1.23	0.14	1.7×10^{-3}
4	20	4	0.16	6.4×10^{-3}	1.0×10^{-5}
6	14.3	2.04	4.2×10^{-2}	8.5×10^{-4}	3.5×10^{-7}
10	9.09	0.83	6.8×10^{-3}	5.6×10^{-5}	3.9×10^{-9}

An application of the formulation embodied in Eq. 7 is given in the following example:

A mill pond of 6 acres, with an average depth of 12 ft, receives settled sewage from a village of 1,800 persons. During a period of low summer flow of 11 cfs the average ultimate BOD of the outlet is 2.42 ppm. The temperature is 68° F, k is 0.44 per day. Assume 0.25 lb per capita per day ultimate BOD.

1. Calculate the short-circuiting parameter N:V = volume of pond = 6(43,560) (12) = 3,130,000 cu ft; T = theoretical detention period = V/Q = (3,130,000)/(11) (86,400) = 3.30 days; L_a = BOD load = (0.25) (1,800) = 450 lb per day; and L_T = BOD in effluent = 2.42 (8.33) (11/1.547) = 144 lb per day.

Then, substituting in Eq. 7 $\frac{L_T}{L_a} = \frac{144}{450} = 0.32 = \left[1 + \frac{(0.44)\ (3.30)}{N} \right]^{-N}$.

By cut-and-try solution, $N = 2.6$.

2. Calculate the BOD in parts per million at the mill pond outlet after construction of a reservoir upstream that is operated so as to increase the summer flow to 25 cfs. The increased dilution will reduce the BOD concentration; this effect, however, will be partly offset by the reduced time of flow. Assume that the value of N will remain the same. T = (3,130,000)/(25) (86,400) = 1.44 days; and $L_T = \frac{450}{\left(1 + \frac{0.44\ (1.44)}{2.6} \right)^{2.6}} = 255$ lb per day. Therefore,

BOD in pond effluent = $\frac{(255)\ (1.547)}{(8.33)\ (25)} = 1.9$ ppm.

While Eqs. 5 and 6 have only an empirical basis, they have been found to formulate the effect of longitudinal mixing in a simple approximate way. A more rational approach is presented in a succeeding section.

Part II—Experimental Technique

Prior to the production of radioisotopes in substantial quantities from atomic piles, tracer tests of flowing liquids were made with salts and dyes. These substances had disadvantages that under some conditions seriously impaired the value of test results. In testing a sedimentation tank, for example, the brine solution introduced at the inlet often would cause density currents that disrupted the normal patterns of flow, and dyes were apt to be partly adsorbed on solid surfaces or else react with other substances in the water to such an extent as to cause a color change. In general both salts and dyes may conveniently be detected in the flow if their concentration exceeds about 1/10 ppm (1/10 milligram per liter), but in many flow systems the mixing is so marked that a considerable proportion of the tracer clouds is dispersed to concentrations appreciably less than the threshold level of 0.1 ppm. The amount of tracer recovered at the downstream station was frequently found to be significantly smaller than that added to the flow. As a consequence, in early tests the so-called actual flow times, obtained by determining the centroid of observed time-concentration curves, were often much smaller than the theoretical average flow times as calculated by Eq. 1 or Eq. 2. From this it was inferred that dead spaces of stationary fluid existed in the flow. Recent tests with more sensitive methods of measuring concentration indicate that such regions do not ordinarily exist and that some mixing occurs in all parts of the flow. The so-called volumetric efficiency (actual flow time divided by theoretical average flow time), therefore, is really a measure of the sensitivity with which concentrations are measured rather than the effective volume of a tank or basin. When sensitive instruments are used, together with suitable tracers—ones that are not adsorbed or changed by reactions with other substances in the water—it has been found that actual flow times approximate the theoretical average time of flow.

Radioisotopes have advantages over the various salts and dyes that have been used as tracers. These may be listed as:

a. They may be detected in extremely small concentrations. A good count (100–1,000 disintegrations per min) may be obtained from samples taken from flows containing as little as 10^{-10} to 10^{-11} mg of radioactive substance per liter. In one test on a pond with a capacity of 18.3 acre ft, only 5 ml of a solution containing 2×10^{-6} grams of radioiodine were used. These extremely low concentrations do not cause observable density currents, so the flow pattern remains undisturbed.

b. The presence of radioisotopes may be readily and positively identified by modern counters of many types. Radioisotopes from natural sources may occasionally be detected in water, but they are normally found in very small amounts and do not interfere with the identification of the tracer. The apparatus for making counts is simple to manipulate and gives precise results.

In small flow systems, hydraulic models, and the like, the passage of the tracer cloud may be detected by radiation (gamma rays) coming through conduit walls so that it is not necessary to sample the flow.

c. No chemical or biochemical reaction occurring in the flow interferes with the rate of disintegration of a radioisotope; temperature changes do not affect the rate of disintegration. Moreover, adsorption of the tracer on solid surfaces can be made a negligibly small factor with a suitable choice of isotope.

The tracer selected for the testing program was radioiodine, I-131. The cost of this radioisotope is not high. One millicurie of I-131 costs $0.75 (1951) and has sufficient activity for a tracer-cloud having a volume of 250,000 gal. I-131 gives off both beta and gamma rays; the latter were found to be sufficiently penetrating so that detection was possible through the walls of small pipes. Iodide is not readily precipitated, adsorbed, or chemically changed by substances normally occurring in water and sewage. Another merit of I-131 was the fact that it yields a linear relationship between concentration and count rate over extensive range of concentrations.

Most important of all, I-131 has a half life of 8 days, which is about ideal for this form of tracer work. The half life of an isotope is the period required for one half of the initial energy to be dissipated. A shipment of 20 millicuries, for example, would at the end of 8 days have a strength of 10 millicuries and at the end of 16 days a strength of 5 millicuries. Elements with long half lives are undesirable because if accidentally spilled or ingested they may be expected to remain active for long periods, giving off radiation that interferes with testing or that may even be dangerous to the health of laboratory personnel. The short-lived elements, on the other hand, are apt to be expensive, since much of their energy may be wasted by disintegration during shipment or delays in testing programs.

One disadvantage pertaining to the use of radioactive tracers as a general research tool is the hazard to the personnel when overexposed to radiation. Excessive exposure may result in cancer, genetic effects, and other damaging results. The guard against excessive radiation must be continual. Precautions include blood tests, the wearing of instruments that record the amount of radiation received, and constant care in handling of radioactive solutions to prevent serious mistakes.

The Atomic Energy Commission[4] has published tolerance limits for various forms of radiation, and organizations using radioisotopes are required to comply with a number of regulations relating to personnel protection. The amounts of radioactive material necessary to perform flow tests of the type described in this paper were small relative to those common in other forms of research, and the safety measures observed were probably more stringent than really necessary.

Experimental Results

To test the use of radioactive tracers in the salt-velocity method in conduits, both field and laboratory experiments were made. Both types of tests were

[4] "Interim Recommendations for the Deposit of Radioactive Wastes by Off-Commission Users," *Isotopes Division Circular B-6*, United States Atomic Energy Commission, Isotopes Division, Oak Ridge, Tenn.

entirely successful, but the laboratory results are perhaps the more interesting and informative. These laboratory tests were performed on a straight length of 2-in. pipe. The radioactive solution was injected quickly into the flow at an upstream point, and points on the curve of the concentration were detected by a

TABLE 2.—RADIOACTIVE TRACER TEST DATA, PIPE LENGTH = 94 FEET

Flow (gallons per second) (1)	Reynolds number (2)	Theoretical time of flow, T (seconds) (3)	Mode (percentage of T) (4)	First trace (percentage of T) (5)	Average time (percentage of T) (6)	Height of mode (7)	Base length (seconds) (8)
0.0205	218	292.6	56.8	46.3	70.9	1.89	135
0.0841	9.410	71.7	82.0	73.6	88.6	....	19
0.1150	10.710	52.4	90.8	82.2	99.4	1.12	18
0.1367	14.210	44.3	98.4	89.4	100.0	0.89	12

Geiger tube placed against the pipe at points along the conduit downstream. Thus by varying the flow rate and the distance from the source, the variation in the cloud-spread could be noted in relation to distance and flow rate.

Tables 2 and 3 illustrate these changes. In Table 2 the variation of the parameters with flow rate is shown. All of these values were obtained at a

TABLE 3.—RADIOACTIVE TRACER DATA, CONSTANT FLOW TESTS

Distance (feet) (1)	Theoretical time of flow, T (seconds) (2)	Mode (percentage of T) (3)	First trace (percentage of T) (4)	Average time (percentage of T) (5)	Height of mode (6)	Base length (seconds) (7)
			(*a*) REYNOLDS NUMBER = 14.210			
21.4	10.2	103.8	73.1	139.0	1.36	....
36.5	17.1	95.4	73.3	101.2	....	....
60.9	28.3	92.9	82.2	108.2	1.28	....
75.6	34.0	93.9	82.5	100.0	1.17	....
94.0	44.3	98.4	89.4	100.0	0.89	....
			(*b*) REYNOLDS NUMBER = 7.490			
31.3	27.8	56.2	45.9	78.6	....	....
44.8	59.4	69.1	55.4	80.1	....	....
53.4	46.9	72.1	63.0	86.0	....	....
78.2	68.7	77.1	69.9	85.2	....	....
			(*c*) REYNOLDS NUMBER = 218			
21.4	69.4	57.7	39.0	84.7	5.30	64
36.5	116.5	60.1	38.6	67.6	2.80	67.5
60.9	195.5	70.3	38.3	60.4	....	86
75.6	219.4	54.8	44.2	57.9	1.06	....
94.0	292.6	56.8	46.3	70.9	....	135

point a constant distance from the injection station. The results show that the mode (time of maximum intensity of radiation), that is given in terms of the theoretical time of passing, increases quite rapidly with the Reynolds number and is thus quite unreliable as a measure of discharge. The mode is often used

in dye studies for this purpose, but the results of these tests show that an error of 20% could be expected. The first trace of the cloud also shows the same unreliable characteristics. The average time of passage seems to be fairly reliable at the higher Reynolds numbers and would seem to be the best measure of discharge of the group.

In Table 3 the average time also stands out as the best measure, although at low Reynolds numbers it too is extremely inaccurate. The variation in the relative height of the mode is also shown, and it is interesting to note the rapid decrease of this quantity as the flow becomes more turbulent.

Results of a tracer test on a rectangular primary settling tank of a sewage treatment plant are shown in Fig. 3. The tank has an average depth of 8.75 ft

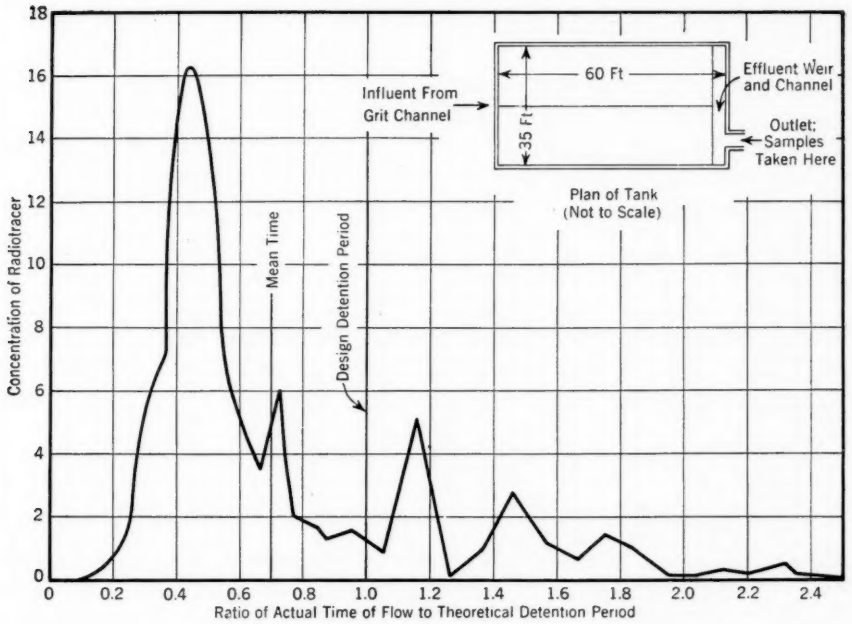

FIG. 3.—CONCENTRATION-TIME DISTRIBUTION FOR A RECTANGULAR PRIMARY SEWAGE SETTLING TANK

and at the time of the test the flow averaged 2.8 mgd. Sewage enters the tank through a series of pipes in parallel and leaves over a 32-ft long weir. A tracer dose of 5.4 millicuries of I-131 was used. The irregularities in the concentration time curve are to be attributed in part to small fluctuations in the rate of flow and in part to inherent irregularities in the tracer cloud. The peak shown at time ratio 1.2 was due to an increase in flow, but the other peaks are the result of normal mixing action. The average time of passage of the tracer (centroid of Fig. 3) was 41 min. The first sample (taken at 10 min) exhibited a significant count. It is of interest to note that dye that had been added at the same time at which the radioisotope was added had traveled only about 25% of the way across the tank toward the outlet at the time the first measurement was made.

The time for the first 10% of the cloud to pass through the tank was 22 min; the last 10% started through at 72 min, and the median flow time was 28.1 min. The N-value (Eq. 5) was found to be 3.0, indicating that a considerable amount of mixing took place.

In Fig. 4 the cumulative time distribution of the tracer cloud is indicated. A typical salt-tracer (sodium chloride) test on another settling tank reported by Charles H. Capen, Jr.,[5] M. ASCE, is presented. The very short flow times shown

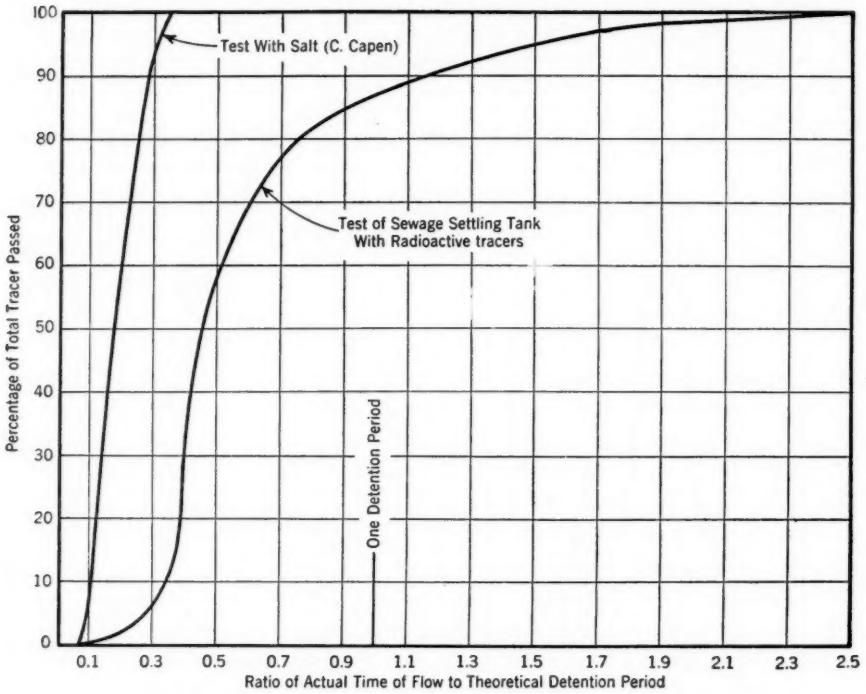

FIG. 4.—CUMULATIVE DISTRIBUTION OF FLOW TIME FOR TRACER CLOUD IN RECTANGULAR SETTLING TANK

on this curve relative to the theoretical average detention time and also relative to the flow times of the other curve obtained with I-131 are noteworthy. This effect is to be attributed mainly to the density currents caused by the addition of brine.

ACKNOWLEDGMENT

The foregoing tests were made at Harvard University, and various places in eastern Massachusetts in connection with an experimental project to investigate the utility of radioactive tracers as a tool of the hydraulician. The project was financed mainly by funds provided by the Boston Society of Civil Engineers. The junior author was awarded a John R. Freeman Fellowship (1948) by this

[5] "Study of Sewage Settling Tank Design," by C. H. Capen, Jr., *Engineering News-Record*, Vol. 99, 1927, p. 833.

organization. Additional aid was furnished by the Sanitary Engineering Division of the Massachusetts Department of Public Health.

Many other flow tests were made in other tanks, including model tanks, pipes, and sewers, and in streams and lakes; these are described in detail in a report by Mr. Archibald[6] to the Freeman Committee.

Appendix—Analysis of the Die-Away Curve in Turbulent Flow

If a waste, that in still water disappears in accordance with Eq. 3, is added at a constant rate at the head of a long stream, the concentration profile after a steady state has been attained will decrease monotonically to zero at infinity downstream. An exact mathematical analysis giving the shape of the concentration profile under general conditions is extremely complex. The following assumptions are made to simplify the formulation and do not greatly lessen the generality of the solution in so far as the mixing caused by eddy diffusion is concerned.

(1) The flow in the stream is steady, uniform, and two dimensional (wide rectangular channel).

(2) The waste does not settle out, nor is it adsorbed on boundaries.

(3) The velocity distribution at a cross section (x-constant) is parabolic,

$$v = v_s - c(h - y)^2 \qquad (8)$$

in which h is the depth of the stream and v_s the surface velocity (the velocity at the bottom of the stream is not necessarily zero). This assumption for the velocity distribution, together with the Prandtl equation

$$\epsilon = \frac{\tau}{\rho \dfrac{dv}{dy}} = \tau_0 \frac{\left(1 - \dfrac{y}{h}\right)}{\rho \dfrac{dv}{dy}} \qquad (9)$$

n which ϵ is the kinematic eddy viscosity and the term $\sqrt{\tau_0/\rho}$ is the shear velocity, leads to the predication of a constant ϵ in the y-direction (normal to the direction of flow):

$$\epsilon = \frac{\tau_0}{2\,c\,h\,\rho} \qquad (10)$$

The turbulence is taken to be isotropic.

(4) The diffusion of substances in suspension or solution in water proceeds exactly as water in water; that is, it is assumed that the rate of transport in a direction s will be $-\epsilon \dfrac{\partial u}{\partial s}$. These assumptions lead to the differential equation

$$\epsilon \frac{\partial^2 u}{\partial x^2} + \epsilon \frac{\partial^2 u}{\partial y^2} - v \frac{\partial u}{\partial x} - k\,u = 0 \qquad (11)$$

[6] "Radioactive Tracers in Flow Tests," by Ralph S. Archibald, *Journal*, Boston Society of Civil Engineers, Vol. 37, 1950, pp. 49–116.

in which ϵ is constant, v is given by Eq. 7, x is the direction of flow (positive downstream), and y is the direction perpendicular to the flow (positive upward).

An appropriate set of boundary conditions are the following:

(*a*) The waste uniformly mixed with the stream flow at a given initial concentration (u_0) enters the channel with a high velocity (as induced by a pump or weir) so that diffusion in the upstream direction against the current is impossible. Then for all y-values at $x = 0$, continuity of transport requires that $u'_0 v - \epsilon \frac{\partial u}{\partial x_{x=0}} = v u_0$ in which u'_0 is the concentration in the channel at $x = 0$ and u_0 is the concentration entering the channel (upstream from the pump)

(*b*) The concentration for all values of y at $x = \infty$ is zero.

(*c*) For all values of x, $\frac{\partial u}{\partial y_{y=0}} = 0$, $\frac{\partial u}{\partial y_{y=h}} = 0$.

Using a method of relaxation, arithmetic integration of Eq. 10 subject to the foregoing boundary conditions, with various values of the parameters c and ϵ has indicated that the former has the greater effect on the concentration profile, particularly upon the order of contact for higher values of x.

An exact solution of Eq. 10 for $c = 0$ (uniform velocity distribution) is given by the following expression:

$$u = \frac{2 u_0}{1 + \sqrt{1 + \frac{4 k \epsilon}{v^2}}} e^{-j} \qquad (12)$$

in which (to simplify typography):

$$j = \frac{\sqrt{1 + 4 k \epsilon / v^2} - 1}{2} \frac{vx}{\epsilon} \qquad (13)$$

The solution for this case depends upon two parameters, k/v and ϵ/v; if these are great, then the profile will differ markedly from that obtained assuming $\epsilon = 0$, namely,

$$u = u_0 e^{-\frac{kx}{v}} \qquad (14)$$

This equation corresponds to Eq. 3.

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